

Chromatic Polynomials of Planar Triangulations, the Tutte Upper Bound, and Chromatic Zeros

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Abstract.

Tutte proved that if G_{pt} is a planar triangulation and $P(G_{pt}, q)$ is its chromatic polynomial, then $|P(G_{pt}, \tau + 1)| \leq (\tau - 1)^{n-5}$, where $\tau = (1 + \sqrt{5})/2$ and n is the number of vertices in G_{pt} . Here we study the ratio $r(G_{pt}) = |P(G_{pt}, \tau + 1)|/(\tau - 1)^{n-5}$ for a variety of planar triangulations. We construct infinite recursive families of planar triangulations $G_{pt,m}$ depending on a parameter m linearly related to n and show that if $P(G_{pt,m}, q)$ only involves a single power of a polynomial, then $r(G_{pt,m})$ approaches zero exponentially fast as $n \rightarrow \infty$. We also construct infinite recursive families for which $P(G_{pt,m}, q)$ is a sum of powers of certain functions and show that for these, $r(G_{pt,m})$ may approach a finite nonzero constant as $n \rightarrow \infty$. The connection between the Tutte upper bound and the observed chromatic zero(s) near to $\tau + 1$ is investigated. We report the first known graph for which the zero(s) closest to $\tau + 1$ is not real, but instead is a complex-conjugate pair. Finally, we discuss connections with nonzero ground-state entropy of the Potts antiferromagnet on these families of graphs.

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1. Introduction

The chromatic polynomial is a function of interest in both graph theory and statistical physics. Let $G = (V, E)$ be a graph with vertex and edge sets V and E , and denote the number of vertices and edges as $n = n(G) = |V|$ and $e(G) = |E|$, respectively. The chromatic polynomial $P(G, q)$ counts the number of ways of assigning q colors to the vertices of G , subject to the condition that no two adjacent vertices have the same color [1, 2]. Such an assignment is called a proper q -coloring of G . The minimum number of colors for a proper q -coloring of a graph G is the chromatic number of G , denoted $\chi(G)$. The connection to statistical physics arises from the fact that $P(G, q) = Z_{PAF}(G, q, 0)$, where $Z_{PAF}(G, q, T)$ denotes the partition function of the Potts antiferromagnet (PAF) on the graph G at temperature T [3]. A further important aspect of this connection involves nonzero ground-state entropy, $S_0 > 0$. This is equivalent to a ground state degeneracy per site $W > 1$, since $S_0 = k_B \ln W$. The Potts antiferromagnet exhibits nonzero ground-state (i.e., zero-temperature) entropy for sufficiently large q on a given graph and serves as a useful model for the study of this phenomenon. The chromatic polynomial of a graph G may be computed via the deletion-contraction relation $P(G, q) = P(G - e, q) - P(G/e, q)$, where $G - e$ denotes G with an edge e deleted and G/e denotes the graph obtained by deleting e and identifying the vertices that it connected. From this follows the general formula $P(G, q) = \sum_{G' \subseteq G} (-1)^{e(G')} q^{k(G')}$, where $G' = (V, E')$ with $E' \subseteq E$ and $k(G')$ denotes the number of connected components of G' .

While the chromatic polynomial $P(G, q)$ enumerates proper q -colorings for positive integer q , it is well-defined for any (real or complex) q . For an arbitrary G and arbitrary q , the calculation of $P(G, q)$ is, in general, an exponentially hard problem. Hence, bounds on $P(G, q)$ are of great value. An interesting upper bound on an evaluation of chromatic polynomials was proved by Tutte. Let us consider a graph G_{pt} that is a planar triangulation (pt), defined as a graph that can be drawn in a plane without any intersecting edges and has the property that all of its faces are triangles. This is necessarily a connected graph. Tutte proved an upper bound on the absolute value of the chromatic polynomial of a planar triangulation, $P(G_{pt}, q)$ evaluated at a certain value of q , viz.,

$$q = \tau + 1 = \frac{3 + \sqrt{5}}{2} = 2.6180339887... \quad (1.1)$$

where $\tau = (1 + \sqrt{5})/2$ is the “golden ratio”, satisfying $\tau + 1 = \tau^2$, or equivalently, $\tau^{-1} = \tau - 1$. Tutte’s upper bound is [4] (see also [5],[6]),

$$|P(G_{pt}, \tau + 1)| \leq U(n(G_{pt})) , \quad (1.2)$$

where

$$U(n) = \tau^{5-n} = (\tau - 1)^{n-5} . \quad (1.3)$$

Since $\tau - 1 < 1$, the upper bound $U(n)$ decreases exponentially rapidly as a function

of n . For a planar triangulation graph G_{pt} , we define the ratio of the evaluation of its chromatic polynomial at $q = \tau + 1$ to the Tutte upper bound as

$$r(G_{pt}) \equiv \frac{|P(G_{pt}, \tau + 1)|}{U(n(G_{pt}))} . \quad (1.4)$$

In this paper we study this ratio $r(G_{pt})$ for various planar triangulation graphs. For this purpose, we shall construct infinite recursive families of planar triangulations $G_{pt,m}$ depending on a parameter m linearly related to n and show that if $P(G_{pt,m}, q)$ only involves a single power of a polynomial, $[\lambda_{G_{pt}}(q)]^m$, then $r(G_{pt,m})$ approaches zero exponentially fast as $m \rightarrow \infty$. We also construct infinite recursive families for which $P(G_{pt,m}, q)$ is a sum of powers of certain functions and show that for these, $r(G_{pt,m})$ may approach a finite nonzero constant as $m \rightarrow \infty$. The Tutte upper bound is sharp, since it is saturated by the simplest planar triangulation, namely the triangle, K_3 ‡. For this graph,

$$P(K_3, \tau + 1) = \tau + 1 , \quad \Rightarrow \quad r(K_3) = 1 . \quad (1.5)$$

However, for planar triangulations with higher n , the upper bound is realized as a strict inequality. For example, for K_4 ,

$$P(K_4, \tau + 1) = -1, \quad \Rightarrow \quad r(K_4) = \tau - 1 = 0.61803.. \quad (1.6)$$

In connection with the upper bound (1.2), Tutte reported his empirical observation that chromatic polynomials of planar triangulations typically have a real zero quite close to $q = \tau + 1$ and remarked that this property could be related to the fact that they obey his bound [4]-[6]. Although several decades have passed since Tutte's work on this topic, the nature of this relation between the zero(s) of a chromatic polynomial $P(G_{pt}, q)$ near to $q = \tau + 1$ and the upper bound (1.2) remains to be understood. We shall elucidate this connection and generalize the investigation to the study of complex zeros of $P(G_{pt}, q)$. In particular, we report the first known case, to our knowledge, of a planar triangulation for which the zero(s) closest to $\tau + 1$ is not a real one, but instead, is a complex-conjugate pair.

Before proceeding, we include some additional notation and remarks. Since a triangulation G_t (whether planar or not) contains at least one triangle, $P(G_t, q)$ always contains the factor $P(K_3, q) = q(q - 1)(q - 2)$. Tutte emphasized the relation of Eq. (1.2) to the golden ratio, τ . It is also of interest to note a relation with certain roots of unity, namely $z_r = e^{\pi i/r}$. One defines $q_r = (z_r + z_r^*)^2 = 4 \cos^2(\pi/r)$, which are often called Tutte-Beraha numbers. The connection with the golden ratio τ is that $q_5 = \tau + 1$.

2. Recursive Families of Planar Triangulations

We have studied various families of planar triangulations that can be constructed in a recursive manner. We denote the m 'th member of such a family as $G_{pt,m}$ and the

‡ The complete graph K_n is defined as the graph with n vertices such that every vertex is connected by an edge to every other vertex. Its chromatic polynomial is $P(K_n, q) = \prod_{s=0}^{n-1} (q - s)$, and its chromatic number is $\chi(K_n) = n$.

number of vertices as

$$n(G_{pt,m}) = \alpha m + \beta , \quad (2.1)$$

where α and β depend on the type of family. Since $U(n) \rightarrow 0$ as $n \rightarrow \infty$ and since m is proportional to n , it follows, first, that for these recursive families of planar triangulations,

$$\lim_{m \rightarrow \infty} P(G_{pt,m}, \tau + 1) = 0 . \quad (2.2)$$

Second, given the upper bound (1.2) and the fact that $U(n)$ approaches zero exponentially fast as $n \rightarrow \infty$, it follows that

$$P(G_{pt,m}, \tau + 1) \text{ approaches zero exponentially fast as } m \rightarrow \infty. \quad (2.3)$$

There are several ways of constructing recursive families of planar triangulations. We have used the following method to construct families for which the chromatic polynomial involves a single power (of a polynomial in q). Start with a basic graph $G_{pt,1}$, drawn in the usual explicitly planar manner. The outer edges of this graph clearly form a triangle, K_3 . Next pick an interior triangle in $G_{pt,1}$ and place a copy of $G_{pt,1}$ in this triangle so that the intersection of the resultant graph with the original $G_{pt,1}$ is the triangle chosen. Denote this as $G_{pt,2}$. Continuing in this manner, one constructs $G_{pt,m}$ with $m \geq 3$. The chromatic polynomial $P(G_{pt,2}, q)$ is calculated from $P(G_{pt,1}, q)$ by using the complete graph intersection theorem. This theorem states that if for two graphs G and H (which are not necessarily planar or triangulations), the intersection $G \cap H = K_p$ for some p , then

$$P(G \cup H, q) = \frac{P(G, q)P(H, q)}{P(K_p, q)} = \frac{P(G, q)P(H, q)}{\prod_{s=0}^{p-1} (q - s)} . \quad (2.4)$$

For our planar triangulations, we use the $p = 3$ special case of this theorem and take advantage of the property that H is a copy of G . Consequently, for planar triangulations formed in this recursive manner, the chromatic polynomial has the form

$$P(G_{pt,m}, q) = P(K_3, q) [\lambda_{G_{pt}}(q)]^m , \quad (2.5)$$

The chromatic number of G_{pt} may be 3 or 4; if $\chi(G_{pt}) = 4$, then $\lambda_{G_{pt}}(q)$ contains the factor $q - 3$. In the case of a planar triangulation which is a strip of the triangular lattice of length m vertices with cylindrical boundary conditions, to be discussed below, an alternate and equivalent way to construct the $(m + 1)$ 'th member of the family is simply to add a layer of vertices to the strip at one end.

We have also devised a method to obtain recursive families of planar triangulations with the property that the chromatic polynomial is a sum of more than one power of a function of q , which we generically write as

$$P(G_{pt,m}, q) = \sum_{j=1}^{j_{max}} c_{G_{pt,j}}(q) [\lambda_{G_{pt,j}}(q)]^m , \quad (2.6)$$

where $j_{max} \geq 2$, the $c_{G_{pt,j}}(q)$ are certain coefficients, and we use the label G_{pt} to refer to the general family of planar triangulations $G_{pt,m}$. We will describe this method in detail

below. Parenthetically, we recall that the form (2.6) is a general one for recursive families of graphs, whether or not they are planar triangulations [7], [8]. For (2.6) evaluated at a given value $q = q_0$, as $m \rightarrow \infty$, and hence $n \rightarrow \infty$, the behavior of $P(G_{pt,m}, q)$ is controlled by which $\lambda_{G_{pt},j}(q)$ is dominant at $q = q_0$, i.e., which of these has the largest magnitude $|\lambda_{G_{pt},j}(q_0)|$. For our purposes, the q_0 of interest is $\tau + 1$. We denote the $\lambda_{G_{pt},j}(q)$ that is dominant at $q = \tau + 1$ as $\lambda_{G_{pt},dom}(\tau + 1)$. Clearly, if $P(G_{pt,m}, q)$ involves only a single power, as in (2.5), then $\lambda_{G_{pt},dom}(\tau + 1) = \lambda_{G_{pt}}(\tau + 1)$.

Thus, in both the cases of Eq. (2.5) and (2.6), a single power $[\lambda_{G_{pt},j}(q)]^m$ dominates the sum as $m \rightarrow \infty$. The upper bound $U(n(G_{pt,m})) = (\tau - 1)^{n-5}$ also has the form of a power, and therefore it is natural to define a (real, non-negative) constant $a_{G_{pt}}$ as

$$a_{G_{pt}} = \lim_{n \rightarrow \infty} [r(G_{pt,m})]^{1/n} = \frac{|\lambda_{G_{pt},dom}(\tau + 1)|^{1/\alpha}}{\tau - 1} . \quad (2.7)$$

To study the asymptotic behavior of $r(G_{pt,m})$ as m and hence n go to infinity, and to determine the resultant constant $a_{G_{pt}}$, we consider the two different cases in turn, namely those in Eqs. (2.5) and (2.6).

We begin with the first case, viz., that of Eq. (2.5). Because $r(G_{pt,m})$ must satisfy the upper bound (1.2) for any m , and because this is satisfied as a strict inequality for all planar triangulations other than the simplest, K_3 , it follows that

$$a_{G_{pt}} < 1 . \quad (2.8)$$

Equivalently,

$$|\lambda_{G_{pt}}(\tau + 1)|^{1/\alpha} < \tau - 1 . \quad (2.9)$$

For all of the $G_{pt,m}$ studied here, $\alpha \geq 1$, so (2.9) implies

$$|\lambda_{G_{pt}}(\tau + 1)| < \tau - 1 . \quad (2.10)$$

From this, we derive two important results, namely that for these recursive families of planar triangulation graphs $G_{pt,m}$ where $P(G_{pt,m}, q)$ has the form (2.5) involving only a single power $[\lambda_{G_{pt}}(q)]^m$, first,

$$\lim_{n \rightarrow \infty} r(G_{pt,m}) = 0 , \quad (2.11)$$

and second,

$$r(G_{pt,m}) \text{ decreases exponentially fast as a function of } m \text{ and } n . \quad (2.12)$$

In contrast, for recursive families of planar triangulation graphs $G_{pt,m}$ where $P(G_{pt,m}, q)$ has the form (2.6) involving two or more powers $[\lambda_{G_{pt},j}(q)]^m$, it is possible for $\lim_{n \rightarrow \infty} r(G_{pt,m})$ to be a nonzero constant (which necessarily lies in the interval $(0, 1)$), so that $a_{G_{pt}} = 1$. This type of behavior means that if one sets $q = \tau + 1$ and takes $n \rightarrow \infty$, the $\lambda_{G_{pt},j}(q)$ that enters in the dominant term in (2.6) satisfies

$$\lambda_{G_{pt},dom}(\tau + 1) = \tau - 1 \quad \text{for } a_{G_{pt}} = 1 . \quad (2.13)$$

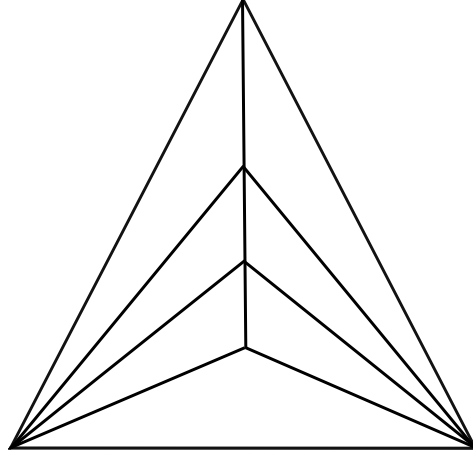


Figure 1. Graph R_4 .

3. The Family $R_m = P_m + P_2$

In this section we illustrate the general results derived in the previous section for recursive families of planar triangulations whose chromatic polynomials have the form (2.5). For this purpose, we consider a family whose m 'th member, denoted R_m , is the join§ of the path graph P_m with $P_2 = K_2$,

$$R_m = P_m + P_2 . \quad (3.1)$$

Thus, $R_1 = K_3$, $R_2 = K_4$, etc. The graph R_4 is shown in Fig. 1. We have $n(R_m) = m + 2$, so $\alpha = 1$ and $\beta = 2$ for this family. An elementary calculation yields $P(R_m, q) = q(q-1)(q-2)[\lambda_R(q)]^{m-1}$, where $\lambda_R(q) = q-3$. Evaluating $P(R_m, q)$ at $q = \tau + 1$, we obtain

$$P(R_m, \tau + 1) = (\tau + 1)(\tau - 2)^{m-1} = \left(\frac{3 + \sqrt{5}}{2} \right) \left(\frac{-3 + \sqrt{5}}{2} \right)^{m-1} \quad (3.2)$$

The ratio of $|P(R_m, \tau + 1)|$ to the Tutte upper bound is

$$r(R_m) = (\tau - 1)^{m-1} = \left(\frac{-1 + \sqrt{5}}{2} \right)^{m-1} \quad (3.3)$$

so that

$$a_R = \frac{-1 + \sqrt{5}}{2} = 0.61803.. \quad (3.4)$$

for this family.

4. The Cylindrical Strip of the Triangular Lattice with $L_y = 3$

Another illustration of recursive families of planar triangulations whose chromatic polynomials have the form (2.5) is provided by the family of cylindrical strips of the

§ The join $G + H$ of two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ is defined as the graph with vertex set $V_{G+H} = V_G \cup V_H$ and edge set E_{G+H} comprised of the union of $E_G \cup E_H$ with the set of edges obtained by connecting each vertex of G with each vertex of H .

triangular (*tri*) lattice. Consider a strip in this family of variable length $L_x = m$ vertices in the x (longitudinal) direction and L_y vertices in the y (transverse) direction. Denote the boundary conditions in the x and y directions as (BC_x, BC_y) , where free and periodic boundary conditions are abbreviated as FBC and PBC . We take the boundary conditions to be (FBC_x, PBC_y) , denoted as cylindrical. The cylindrical strip of the triangular lattice with $L_y = 3$, i.e., transverse cross sections consisting of triangles, and arbitrary length $L_x = m$, denoted TC_m , is a planar triangulation, with $n(TC_m) = 3m$ vertices. In this family, TC_1 is a degenerate case of a triangle, K_3 , while TC_2 is the graph of the octahedron. TC_m may also be constructed in the recursive manner described in Sect. 2 by starting with TC_2 , choosing an interior triangle, and placing a copy of TC_2 onto this triangle to get TC_3 , and so forth for $m \geq 4$. Thus, TC_m may also be considered as a recursively iterated octahedron graph. An elementary calculation yields $P(TC_m, q) = q(q-1)(q-2) [\lambda_{TC}(q)]^{m-1}$, where $\lambda_{TC}(q) = q^3 - 9q^2 + 29q - 32$. Evaluating $P(TC_m, q)$ at $q = \tau + 1$ gives

$$P(TC_m, \tau + 1) = \left(\frac{3 + \sqrt{5}}{2} \right) (-11 + 5\sqrt{5})^{m-1}. \quad (4.1)$$

Hence,

$$r(TC_m) = \frac{(3 + \sqrt{5})}{4} (3 - \sqrt{5})^m, \quad (4.2)$$

so that

$$a_{TC} = (3 - \sqrt{5})^{1/3} = 0.91415... \quad (4.3)$$

$P(TC_m, q)$ has a real zero near to $\tau + 1$, at $q \simeq 2.546602$.

5. Iterated Icosahedron Graphs

A more complicated family of planar triangulations yielding chromatic polynomials of the form (2.5) is provided by recursive iterates of the icosahedron graph I_1 , shown in Fig. 2. We construct these recursive iterates I_m with $m \geq 2$ by the procedure given in Sect. 2. The graph I_m has $n(I_m) = 3 + 9m$ vertices. The chromatic polynomial is

$$P(I_m, q) = q(q-1)(q-2) [f_I(q)]^m \quad (5.1)$$

where

$$\begin{aligned} f_I(q) = & (q-3)(q^8 - 24q^7 + 260q^6 - 1670q^5 + 6999q^4 - 19698q^3 \\ & + 36408q^2 - 40240q + 20170). \end{aligned} \quad (5.2)$$

We calculate

$$P(I_m, \tau + 1) = \left(\frac{3 + \sqrt{5}}{2} \right) \left(\frac{-23955 + 10713\sqrt{5}}{2} \right)^m. \quad (5.3)$$

Hence,

$$r(I_m) = \frac{(3 + \sqrt{5})}{4} \left(\frac{-315 + 141\sqrt{5}}{2} \right)^m, \quad (5.4)$$

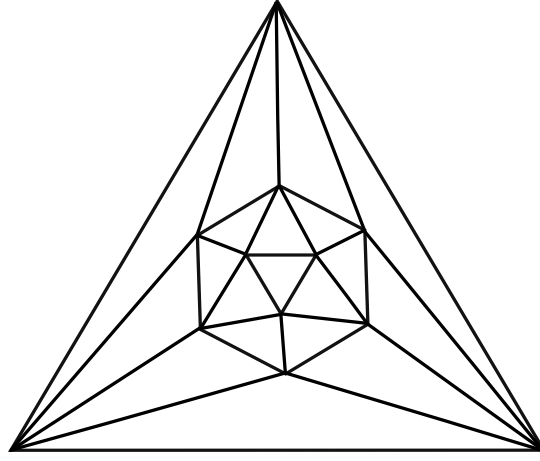


Figure 2. Icosahedron graph, I_1 .

so that

$$a_I = \left(\frac{-315 + 141\sqrt{5}}{2} \right)^{1/9} = 0.80552... \quad (5.5)$$

$P(I_m, q)$ has a real zero quite close to $\tau + 1$, at $q \simeq 2.6181973$ and another real zero near to q_7 , at $q \simeq 3.222458$.

6. The Family B_n

In this section we consider a recursive family of planar triangulations whose chromatic polynomials have the form (2.6) with $j_{max} = 3$. This is the bipyramid family, which is the join

$$B_n = \bar{K}_2 + C_{n-2} \quad (6.1)$$

for $n \geq 5$, where \bar{K}_p , the complement of K_p , is the graph of p disjoint vertices with no edges. Clearly, $n(B_n) = n$. For illustration, the graphs B_7 and B_8 are shown in Figs. 3 and 4. In Figs. 3 and 4, the uppermost and the lower middle vertices of B_n have degree $n - 2$, so that the degrees of these two vertices go to infinity as $n \rightarrow \infty$. All of the other vertices have degree 4. The chromatic polynomial for this graph is of the form (2.6) with three λ s:

$$P(B_n, q) = \sum_{j=1}^3 c_{B,j}(q) [\lambda_{B,j}(q)]^{n-2}, \quad (6.2)$$

where

$$c_{B,j}(q) = q, \quad c_{B,2}(q) = q(q-1), \quad c_{B,3}(q) = q(q^2 - 3q + 1) \quad (6.3)$$

and

$$\lambda_{B,1}(q) = q - 2, \quad \lambda_{B,2}(q) = q - 3, \quad \lambda_{B,3}(q) = -1. \quad (6.4)$$

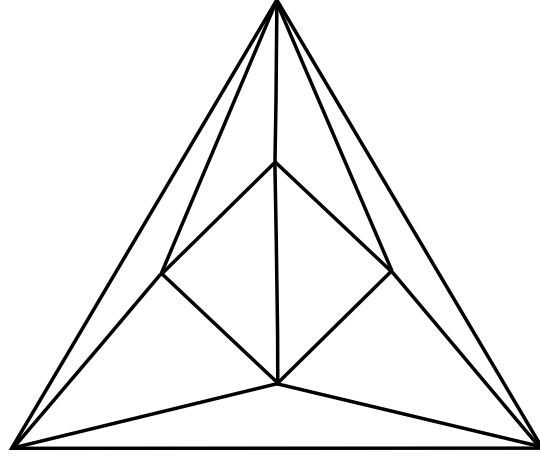


Figure 3. Graph B_7 .

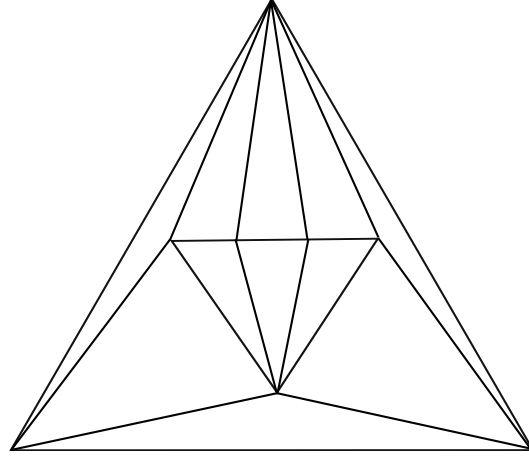


Figure 4. Graph B_8 .

$P(B_n, q)$ contains the factor $P(K_3, q)$ if n is even and $P(K_4, q)$ if n is odd; related to this, $\chi(B_n) = 3$ if n is even and $\chi(B_n) = 4$ if n is odd.

Evaluating $P(B_n, q)$ at $q = \tau + 1$, we find

$$P(B_n, \tau + 1) = (\tau + 1) \left[(\tau - 1)^{n-2} + \tau(\tau - 2)^{n-2} \right]. \quad (6.5)$$

Note that $c_{B,3}(q)$ vanishes for $q = \tau + 1$ (see the related Eq. (2.8) of [12]), so that the $j = 3$ term does not contribute for this value of q . Hence,

$$r(B_n) = (\tau - 1) \left[1 + \tau(1 - \tau)^{n-2} \right] \quad (6.6)$$

Since $|1 - \tau| < 1$, the second term, $\tau(1 - \tau)^{n-2}$, vanishes as $n \rightarrow \infty$, so

$$\lim_{n \rightarrow \infty} r(B_n) = \tau - 1 = \frac{-1 + \sqrt{5}}{2} = 0.61803.. \quad (6.7)$$

and

$$a_B = 1. \quad (6.8)$$

Because $1 - \tau$ is negative, the ratios $r(B_n)$ form a sequence such that for increasing even (odd) n , $r(B_n)$ approaches the $n \rightarrow \infty$ limit $\tau - 1$ from above (below). For $n \geq 6$, $P(B_n, q)$ has real zeros that are close to $q = \tau + 1$. These form a sequence such that for even (odd) n the nearby zero is slightly less than (greater than) $q = \tau + 1$, respectively. These are listed in Table 1 for n from 6 to 20.

Table 1. Location of zero q_z of $P(B_n, q)$ closest to $\tau + 1 = 2.6180339887..$ for n from 6 to 20. Notation $ae-n$ means $a \times 10^{-n}$.

n	q_z	$q_z - (\tau + 1)$
6	2.546602	-0.07143
7	2.677815	0.05978
8	2.594829	-0.02321
9	2.636118	0.01808
10	2.609130	-0.8904e-2
11	2.624356	0.6322e-2
12	2.614541	-3.493e-3
13	2.620356	2.322e-3
14	2.616673	-1.361e-3
15	2.618905	0.8713e-3
16	2.617509	-0.5254e-3
17	2.618364	3.301e-4
18	2.617832	-2.017e-4
19	2.618160	1.256e-4
20	2.617957	-0.7725e-4

The continuous accumulation set of zeros of $P(B_n, q)$ in the complex q plane as $n \rightarrow \infty$ was studied in [10, 8]. These form curves defined by the equality in magnitude of the dominant $\lambda_{B,j}$, in accordance with general results for recursive functions [11]. These curves extend infinitely far from the origin if such an equality can be satisfied as $|q| \rightarrow \infty$, as was discussed in [8, 13, 14]. Here, because the equality $|\lambda_{B,1}| = |\lambda_{B,2}|$, i.e., $|q - 2| = |q - 3|$, holds for the infinite line $\text{Re}(q) = 5/2$, part of the boundary \mathcal{B} extends infinitely far away from the origin in the q plane, i.e., passes through the origin of the $1/q$ plane. The locus \mathcal{B} separates the q plane into three regions (see Fig. 2 and Eqs. (4.11)-(4.16) in [8]): (i) region R_1 , which includes the semi-infinite interval $q > 3$ on the real axis; (ii) region R_2 , which includes the semi-infinite interval $q < 2$ on the real axis; and (iii) region R_3 , a region bounded below on the real axis by $q = 2$ and above by $q = 3$, and extending upward and downward to the triple points $q = (5 \pm \sqrt{3}i)/2$, where the three regions are contiguous. Denoting q_c as the maximal point where \mathcal{B} intersects the real- q axis, we have $q_c = 3$ here. The boundary between regions R_1 and R_2 for $|\text{Im}(q)| > \sqrt{3}/2$ is comprised by semi-infinite segments of the vertical line $\text{Re}(q) = 5/2$. In region R_1 , $\lambda_{B,1}$ is dominant.

In the calculation of the constant $a_{G_{pt}}$ of Eq. (2.7), we note that one first sets $q = \tau + 1$ and then takes $n \rightarrow \infty$. In general, for a special set of values of q , denoted

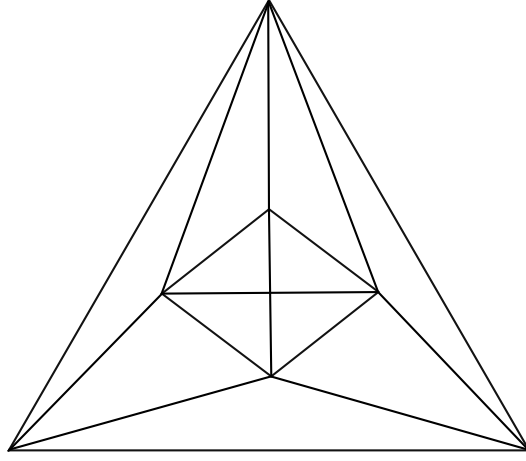


Figure 5. Graph H_8 .

$\{q_s\}$, one has the noncommutativity [8, 9]

$$\lim_{q \rightarrow q_s} \lim_{n \rightarrow \infty} [P(G, q)]^{1/n} \neq \lim_{n \rightarrow \infty} \lim_{q \rightarrow q_s} [P(G, q)]^{1/n} . \quad (6.9)$$

This noncommutativity is illustrated by the family B_n , for which the set of special values $\{q_s\}$ includes $q = 0, 1, 2, 3$, and $\tau + 1$. Thus, for the evaluation of a_B , one first sets $q = \tau + 1$ (which is a point lying in region R_3) and then takes $n \rightarrow \infty$. In this case, since $c_{B,3}(q) = 0$, for $q = \tau + 1$, the $(\lambda_{B,3})^{n-2}$ term is multiplied by zero and does not contribute to the chromatic polynomial, and the resultant dominant term is $\lambda_{B,1} = q - 2$, whence the results in Eqs. (6.6) and (6.8). In contrast, if one takes $n \rightarrow \infty$ first, then the dominant term in region R_3 is $\lambda_{B,3} = -1$ (and the dominant term in R_2 is $q - 3$).

7. The Family H_n

For comparative purposes, it is valuable to study another recursive family of planar triangulations with chromatic polynomials of the multi-term form (2.6). We denote this family as H_n , which is well-defined for $n \geq 8$ and has $n(H_n) = n$. In Fig. 5 we show the lowest member of this family, H_8 . The next higher member, H_9 , is constructed by adding a vertex and associated edges in the central “diamond”, as shown in Fig. 6, and so forth for higher members.

For this family of planar triangulations we calculate the chromatic polynomial as

$$P(H_n, q) = \sum_{j=1}^3 c_{H,j}(q) [\lambda_{H,j}(q)]^{n-5} , \quad (7.1)$$

where

$$c_{H,1}(q) = q(q - 3)^3 \quad (7.2)$$

$$c_{H,2}(q) = q(q - 1)(q^3 - 9q^2 + 30q - 35) \quad (7.3)$$

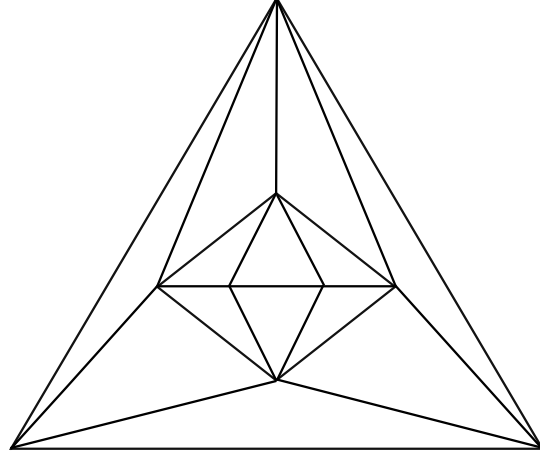


Figure 6. Graph H_9 .

and

$$c_{H,3}(q) = -q(q-3)(q-5)(q^2-3q+1) \quad (7.4)$$

Thus, $j_{max} = 3$ for this family, as was the case for the B_n family. The $\lambda_{H,j}$ are also the same as those for the B_n family, namely

$$\lambda_{H,1}(q) = q-2, \quad \lambda_{H,2}(q) = q-3, \quad \lambda_{H,3}(q) = -1. \quad (7.5)$$

$P(H_n, q)$ contains the factor $P(K_4, q)$ and hence has $\chi(H_n) = 4$.

Evaluating $P(H_n, q)$ at $q = \tau + 1$, we find

$$P(H_n, \tau + 1) = \left(\frac{-7 + 3\sqrt{5}}{2} \right) (\tau - 1)^{n-5} + \left(\frac{5 - 3\sqrt{5}}{2} \right) (\tau - 2)^{n-5}. \quad (7.6)$$

Consequently,

$$r(H_n) = \frac{-7 + 3\sqrt{5}}{2} + \left(\frac{5 - 3\sqrt{5}}{2} \right) \left(\frac{1 - \sqrt{5}}{2} \right)^{n-5}. \quad (7.7)$$

Since $|(1 - \sqrt{5})/2| < 1$, the second term in Eq. (7.7) vanishes as $n \rightarrow \infty$, so

$$\lim_{n \rightarrow \infty} r(H_n) = \frac{7 - 3\sqrt{5}}{2} = 0.145898.. \quad (7.8)$$

and

$$a_H = 1. \quad (7.9)$$

8. Nonzero Ground-State Entropy

In this section we discuss some properties of these families of planar triangulations relevant for statistical physics. We recall that the entropy per vertex is given by $S = k_B \ln W$, where W is the degeneracy per vertex, related to the total degeneracy of configurations W_{tot} by $W = \lim_{n \rightarrow \infty} (W_{tot})^{1/n}$. Let us denote the formal limit of a family of graphs G as $n(G) \rightarrow \infty$ by the symbol $\{G\}$. The total ground-state entropy

of the Potts antiferromagnet on the graph G is $W_{tot}(G, q) = Z_{PAF}(G, q, T = 0) = P(G, q)$, so that, formally, $W(\{G\}, q) = \lim_{n \rightarrow \infty} [P(G, q)]^{1/n}$. However, owing to the noncommutativity (6.9) [8], it is necessary to specify the order of limits in this formal expression. For a particular value $q = q_s$, we thus define

$$W_{qn}(\{G\}, q_s) = \lim_{q \rightarrow q_s} \lim_{n \rightarrow \infty} [P(G, q)]^{1/n} \quad (8.1)$$

and

$$W_{nq}(\{G\}, q_s) = \lim_{n \rightarrow \infty} \lim_{q \rightarrow q_s} [P(G, q)]^{1/n} . \quad (8.2)$$

For real $q \geq \chi(G_{pt,m})$, both of these definitions are equivalent, and in this case we shall write $W_{qn}(\{G\}, q_s) = W_{nq}(\{G\}, q_s) \equiv W(\{G\}, q_s)$. We proceed to discuss these quantities for the families of graphs studied here.

For the family R_m with $\chi(R_m) = 4$, we have, for $q \geq 4$,

$$W(\{R\}, q) = q - 3 . \quad (8.3)$$

Since $W(\{R\}, q) > 1$ for $q > 4$, it follows that in this range of q the Potts antiferromagnet exhibits nonzero ground-state entropy $S_0 = k_B \ln(q - 3)$ on the $m \rightarrow \infty$ limit of this family of graphs.

For the family TC_m of cylindrical strips of the triangular lattice, with $\chi(TC_m) = 3$, we have, for real $q \geq 3$,

$$W(\{TC\}, q) = [\lambda_{TC}(q)]^{1/3} . \quad (8.4)$$

This function has the property that $W(\{TC\}, 3) = 1$ and $W(\{TC\}, q) > 1$ for real $q > 3$, so that the Potts AF has nonzero ground-state entropy, i.e., $S_0 = (1/3)k_B \ln[\lambda_{TC}(q)] > 0$, for $q > 3$ on the infinite-length limit of this lattice strip. Parenthetically, we note that many strips of the triangular lattice, such as those with free, cyclic, or Möbius boundary conditions with more than one K_3 , or with toroidal boundary conditions [15]-[18] are not planar triangulations, since they are either not planar or, if planar, they contain at least one face that is not a triangle, and hence their chromatic polynomials are not subject to the Tutte bound.

For the family of iterated icosahedra, I_m , $\chi(I_m) = 4$ and, for $q > 3$, we calculate

$$W_{qn}(\{I\}, q) = [(q - 3)f_I(q)]^{1/9} . \quad (8.5)$$

Now $W(\{I\}, q) > 1$ for $q > 3.5133658\dots$, so that $S_0 = (1/9)k_B \ln[(q - 3)f_I(q)]$ is positive in this interval. Note that $W(\{I\}, 4) = (10)^{1/9} = 1.29155\dots$

For the bipyramid family B_n , with $\chi(B_n) = 3$ for even n and $\chi(B_n) = 4$ for odd n and for the family H_n , with $\chi(H_n) = 4$, we have $W_{qn}(\{G\}, q) = q - 2$, and $W_{qn}(\{G\}, q) = W_{nq}(\{G\}, q)$ for $q \geq 4$, where $\{G\} = \{B\}$ or $\{H\}$, whence

$$W(\{G\}, q) = q - 2 \quad \text{if } q \geq 4 \quad \text{for } \{G\} = \{B\} \text{ or } \{H\} . \quad (8.6)$$

Thus, with the definition $W_{qn}(\{G\}, q)$, there is nonzero ground-state entropy $S_0 = k_B \ln(q - 2)$ for $q > 3$ and with either definition, $W(\{G\}, q) > 1$ and $S_0 > 0$ for $q \geq 4$ for $\{G\} = \{B\}$ or $\{H\}$.

9. On the Chromatic Zero Nearest to $\tau + 1$

We have calculated and analyzed the chromatic polynomials of a number of other planar triangulation graphs. We construct these graphs by starting with a triangle, K_3 , inserting a circuit graph with ℓ vertices, C_ℓ , inside it, connecting the vertices of this circuit graph in an appropriate manner with the three outer vertices of the original triangle, and then, for $\ell \geq 4$, tiling the interior of the C_ℓ with triangles. We have also studied the zeros of the chromatic polynomials of these planar triangulations, i.e., the chromatic zeros. Before reporting our new result, we recall some properties of real chromatic zeros. For a general graph G , it is elementary that there are no negative chromatic zeros. Further, there are no chromatic zeros in the intervals $(0, 1)$ or $(1, 32/27]$ [19]. For a planar triangulation G_{pt} , there are no zeros in the interval $(2, 2.546602\dots)$, where the second number is the unique real zero of $\lambda_{TC}(q)$ [20]. In 1992, Woodall conjectured that a planar triangulation has no chromatic zeros in the interval $(q_m, 3)$, where $q_m = 2.6778146\dots$ is the unique real zero of $q^3 - 9q^2 + 30q - 35$ [20], but in 2005 he reported that he had found a counterexample to his conjecture. We have also found counterexamples, which we shall discuss below. The four-color theorem states that for a planar graph, G_p , $P(G_p, 4) > 0$. It has been shown that for a particular family of planar graphs given in [22], there is an accumulation of real zeros approaching $q = 4$ from below [23].

It may be recalled that Tutte noticed that many planar triangulations have real chromatic zeros close to $q = \tau + 1$. We have investigated the following question: if G_{pt} is a planar triangulation, is it true that the chromatic zero of G_{pt} nearest to $\tau + 1$ is always real? This is true for almost all of the families that we have studied; however, we have shown that it is not true in general.

We have found a planar triangulation which is, to our knowledge, the first case for which the zero closest to $\tau + 1$ is not real but instead the zeros closest to $\tau + 1$ form a complex-conjugate pair. We denote this graph as $G_{CM,m}$ and show a picture of the $m = 1$ case in Fig. 7. The graph $G_{CM,m}$ has $n(G_{CM,m}) = 3 + 8m$. The chromatic polynomial of this graph has the form (2.5). We calculate

$$P(G_{CM,m}, q) = q(q-1)(q-2) [\lambda_{CM}(q)]^m, \quad (9.1)$$

where

$$\begin{aligned} f_{CM}(q) = & q^8 - 24q^7 + 259q^6 - 1641q^5 + 6674q^4 - 17818q^3 + 30418q^2 \\ & - 30250q + 13360. \end{aligned} \quad (9.2)$$

In addition to the zeros at $q = 0, 1, 2$, $P(G_{CM,m}, q)$ has eight other zeros, which form four complex-conjugate pairs. One of these pairs, namely the zeros at $2.641998 \pm 0.014795i$, lie closest to $\tau + 1$, each one being a distance $0.028163\dots$ from this point. For reference,

$$r(G_{CM,m}) = \left(\frac{115 - 51\sqrt{5}}{2} \right)^m. \quad (9.3)$$

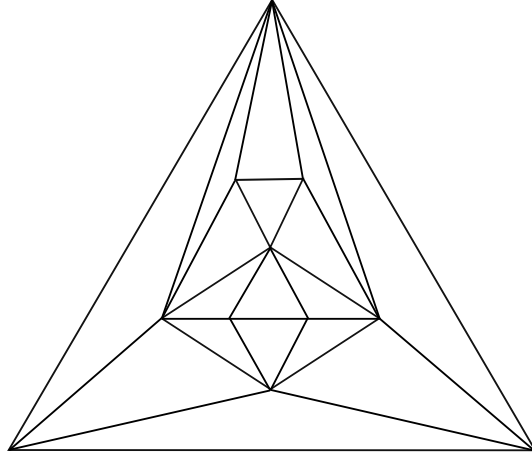


Figure 7. Graph $G_{CM,1}$.

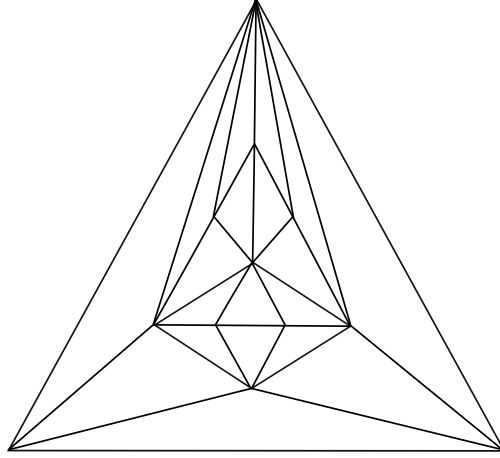


Figure 8. Graph G_{ce12} .

Hence,

$$a_{G_{CM}} = \left(\frac{115 - 51\sqrt{5}}{2} \right)^{1/8} = 0.885185.. \quad (9.4)$$

We have found a number of counterexamples to the 1992 Woodall conjecture that for a planar triangulation, there are no chromatic zeros in the interval $(q_m, 3)$. We show one in Fig. 8. For this graph we calculate the chromatic polynomial

$$\begin{aligned} P(G_{ce12}, q) = & q(q-1)(q-2)(q-3)(q^8 - 24q^7 + 260q^6 - 1658q^5 \\ & + 6800q^4 - 18337q^3 + 31668q^2 - 31915q + 14314) . \end{aligned} \quad (9.5)$$

Besides its zeros at $q = 0, 1, 2, 3$, this chromatic polynomial has two more real zeros, at $q = 2.61461437..$ and $q = 2.81889716...$. The former of these is close to $\tau + 1$, and the latter is notable as lying in the interval $(2.677814..., 3)$. The other zeros of $P(G_{ce12}, q)$

form three complex-conjugate pairs. We note that

$$r(G_{ce12}) = 101 - 45\sqrt{5} = 0.376941... \quad (9.6)$$

10. Conclusions

In this paper we have studied the ratio of the chromatic polynomial of a planar triangulation evaluated at $q = \tau + 1$ to the Tutte upper bound, $r(G_{pt}) = |P(G_{pt}, \tau + 1)|/(\tau - 1)^{n-5}$, for a variety of G_{pt} graphs. We have constructed and analyzed infinite recursive families of planar triangulations $G_{pt,m}$ depending on a parameter m linearly related to n and have shown that if $P(G_{pt,m}, q)$ only involves a single power of a polynomial, $[\lambda_{G_{pt}}(q)]^m$, then $r(G_{pt,m})$ approaches zero exponentially fast as $n \rightarrow \infty$. We have analyzed infinite recursive families for which $P(G_{pt,m}, q)$ is a sum of several powers, as in Eq. (2.6) with $j_{max} \geq 2$ and have shown that for these families, $r(G_{pt,m})$ may approach a finite nonzero constant as $m \rightarrow \infty$. We have also elucidated the connection between the Tutte upper bound and chromatic zero(s) near to $\tau + 1$. In particular, we have discovered a graph which is, to our knowledge, the first one for which the zero(s) closest to $\tau + 1$ is not real, but instead is a complex-conjugate pair. In a final section, we have discussed interesting connections with ground-state entropy of the Potts antiferromagnet on the $n \rightarrow \infty$ limits of these families of graphs.

11. Acknowledgments

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